

On the heat equation with nonlinearity and singular anisotropic potential on the boundary

Marcelo F. de Almeida

Universidade Federal de Sergipe, Departamento de Matemática,

CEP 49100-000, Aracaju-SE, Brazil.

E-mail:nucaltiado@gmail.com

Lucas C. F. Ferreira *

Universidade Estadual de Campinas, Departamento de Matemática,

CEP 13083-859, Campinas-SP, Brazil.

E-mail:lcff@ime.unicamp.br

Juliana C. Precioso

Unesp-IBILCE, Departamento de Matemática,

CEP 15054-000, São José do Rio Preto-SP, Brazil.

E-mail:precioso@ibilce.unesp.br

Abstract

This paper concerns with the heat equation in the half-space \mathbb{R}_+^n with nonlinearity and singular potential on the boundary $\partial\mathbb{R}_+^n$. We develop a well-posedness theory (without using Kato and Hardy inequalities) that allows us to consider critical potentials with infinite many singularities and anisotropy. Motivated by potential profiles of interest, the analysis is performed in weak L^p -spaces in which we prove key linear estimates for some boundary operators arising from the Duhamel integral formulation in \mathbb{R}_+^n . Moreover, we investigate qualitative properties of solutions like self-similarity, positivity and symmetry around the axis $\overrightarrow{Ox_n}$.

AMS MSC2010: 35K05, 35A01, 35K20, 35B06, 35B07, 35C06, 42B35

Keywords: Heat equation, Singular potentials, Nonlinear boundary conditions, Self-similarity, Symmetry, Lorentz spaces.

*L. Ferreira was supported by FAPESP and CNPQ, Brazil. (corresponding author)

1 Introduction

Heat equations with singular potentials have attracted the interest of many authors since the work of Baras and Goldstein [8] in the 80's. In a smooth domain $\Omega \subset \mathbb{R}^n$, they studied the Cauchy problem for the linear heat equation

$$u_t - \Delta u - V(x)u = 0 \quad (1.1)$$

with singular potential

$$V(x) = \frac{\lambda}{|x|^2} \quad (1.2)$$

and obtained a threshold value $\lambda_* = \frac{(n-2)^2}{4}$ with $n \geq 3$ for existence of positive L^2 -solutions. The potential in (1.2) is called inverse square (Hardy) potential and is an example of potential arising from negative power laws. This class of potentials appears in a number of physical phenomena (see e.g. [15],[16],[22],[30],[32],[35] and references therein) and can be classified according to the number of singularities (poles), σ -degree of the singularity (order of the poles), dependence on directions (anisotropy) and decay at infinity. One of the most difficult cases is the one of anisotropic critical potentials, namely

$$V(x) = \sum_{i=1}^l \frac{v_i \left(\frac{x-x^i}{|x-x^i|} \right)}{|x-x^i|^\sigma}, \quad (1.3)$$

where $v_i(z) \in BC(\mathbb{S}^{n-1})$, $x^i \in \overline{\Omega}$, $l \in \mathbb{N} \cup \{\infty\}$, and the parameter σ is the order of the poles $\{x^i\}_{i=1}^l$. The potential is called isotropic (resp. anisotropic) when the v_i 's are independent (resp. dependent) of the directions $\frac{x-x^i}{|x-x^i|}$, that is, they are constant. In the case $l = 1$ (resp. $l > 1$), V is said to be monopolar (resp. multipolar). The criticality means that σ is equal to order of the PDE inside the domain or of the boundary condition, according to the type of problem considered. The critical case introduces further difficulties in the mathematical analysis of the problem because Vu cannot be handled as a lower order term (see [15]). Examples of (1.3) are

$$V(x) = \sum_{i=1}^l \frac{\lambda_i}{|x-x^i|^\sigma} \text{ and } V(x) = \sum_{i=1}^l \frac{(x-x^i) \cdot d^i}{|x-x^i|^{\sigma+1}}, \quad (1.4)$$

where $x^i = (x_1^i, x_2^i, \dots, x_n^i)$ and $d^i \in \mathbb{R}^n$ are constant vectors. In the theory of Schrodinger operators, the potentials in (1.4) are called multipolar Hardy potentials and multiple dipole-type potentials, respectively (see [15] and [16]).

In this paper, we consider a nonlinear counterpart for (1.1) in the half-space with critical singular boundary potential, which reads as

$$\partial_t u = \Delta u \text{ in } \Omega, \ t > 0 \quad (1.5)$$

$$\partial_n u = h(u) + V(x')u \text{ in } \partial\Omega, \ t > 0 \quad (1.6)$$

$$u(x, 0) = u_0(x), \text{ in } \Omega, \quad (1.7)$$

where $\Omega = \mathbb{R}_+^n$, $n \geq 3$ and $\partial_n = -\partial_{x_n}$ stands for the normal derivative on $\partial\mathbb{R}_+^n$. For the nonlinear term, we assume that the function $h : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $h(0) = 0$ and

$$|h(a) - h(b)| \leq \eta |a - b| (|a|^{\rho-1} + |b|^{\rho-1}), \quad (1.8)$$

where $\rho > 1$ and the constant η is independent of $a, b \in \mathbb{R}$. A classical example of h satisfying these conditions is $h(u) = \pm |u|^{\rho-1} u$.

Our goal is to develop a global-in-time well-posedness theory for (1.5)-(1.7), under smallness conditions on certain weak norms of u_0, V , that allows to consider critical potentials on the boundary with infinite many singularities. For that matter, we employ the framework of weak- L^p spaces (i.e., $L^{(p,\infty)}$ -spaces) and take $V \in L^{(n-1,\infty)}(\partial\mathbb{R}_+^n)$. Since $L^p(\partial\mathbb{R}_+^n)$ contains only trivial homogeneous functions, a motivation naturally appears for considering weak- L^p spaces. In fact, due to Chebyshev's inequality, we have the continuous inclusion $L^p(\partial\mathbb{R}_+^n) \subset L^{(p,\infty)}(\partial\mathbb{R}_+^n)$ and then $L^{(p,\infty)}$ can be regarded as a natural extension of L^p which contains homogeneous functions of degree $\sigma = -(n-1)/p$. The critical case for (1.5)-(1.7) with potential (1.3) corresponds to $\sigma = 1$ (so, $p = n-1$) and we have that

$$\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}_+^n)} \leq \sum_{i=1}^l \sup_{x' \in \mathbb{S}^{n-2}} |v_i(x')| \left\| \frac{1}{|x' - x^i|} \right\|_{L^{(n-1,\infty)}(\partial\mathbb{R}_+^n)} \leq C \sum_{i=1}^l \sup_{x' \in \mathbb{S}^{n-2}} |v_i(x')|, \quad (1.9)$$

where

$$C = \left\| |x'|^{-1} \right\|_{L^{(n-1,\infty)}(\partial\mathbb{R}_+^n)} < \infty. \quad (1.10)$$

Of special interest is when the set $\{x^i\}_{i=1}^l \subset \partial\mathbb{R}_+^n$ and so V has a number l of singularities on the boundary which can be infinite provided that the infinite sum in (1.9) is finite.

We address (1.5)-(1.7) by means of the following equivalent integral formulation

$$u(x, t) = \int_{\mathbb{R}_+^n} G(x, y, t) u_0(y) dy + \int_0^t \int_{\partial\mathbb{R}_+^n} G(x, y', t-s) [h(u) + Vu](y', s) dy' ds \quad (1.11)$$

where $G(x, y, t)$ is the heat fundamental solution in \mathbb{R}_+^n given by

$$G(x, y, t) = (4\pi t)^{-\frac{n}{2}} \left[e^{-\frac{|x-y|^2}{4t}} + e^{-\frac{|x-y^*|^2}{4t}} \right], \ x, y \in \overline{\mathbb{R}_+^n}, \ t > 0, \quad (1.12)$$

with $y^* = (y', -y_n)$ and $y' = (y_1, \dots, y_{n-1}) \in \partial\mathbb{R}_+^n$. Here solutions for (1.11) are looked for in $BC((0, \infty); \mathcal{X}_{p,q})$ where $\mathcal{X}_{p,q}$ is a suitable Banach space that can be identified with $L^{(p,\infty)}(\mathbb{R}_+^n) \times L^{(q,\infty)}(\partial\mathbb{R}_+^n)$. The norm in $\mathcal{X}_{p,q}$ provides a $L^{(q,\infty)}$ -information for $u|_{\partial\mathbb{R}_+^n}$ without assuming any positive regularity condition on u . Notice that this space is specially useful in order to treat singular boundary terms like (1.3). L^r -versions of $\mathcal{X}_{p,q}$ (i.e. $L^{r_1}(\Omega) \times L^{r_2}(\partial\Omega)$) was employed in [36] and [20] for studying weak solutions for an elliptic and parabolic PDE in bounded domains Ω , respectively. Let us observe that $\| |x'|^{-1} \|_{L^{r_2}(\partial\mathbb{R}_+^n)} = \infty$ for all $1 \leq r_2 \leq \infty$ (compare with (1.10)) which prevents the use of the spaces of [20, 36] for our purposes.

Furthermore, we investigate qualitative properties of solutions like positivity, symmetries (e.g. invariance around the axis $\overrightarrow{Ox_n}$) and self-similarity, under certain conditions on $u_0, V, h(\cdot)$. For the latter, the indexes of spaces are chosen so that their norms are invariant by scaling of (1.5)-(1.6) (see (3.1) below), namely $p = n(\rho - 1)$ and $q = (n - 1)(\rho - 1)$.

Common tools used to handle (1.1) and (1.5)-(1.6) with critical potentials are the Hardy and Kato inequalities, which read respectively as

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}^2, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n) \quad (1.13)$$

$$2 \frac{\Gamma(\frac{n}{4})^2}{\Gamma(\frac{n-2}{4})^2} \int_{\partial\mathbb{R}_+^n} \frac{u^2}{|x|} dx \leq \|\nabla u\|_{L^2(\mathbb{R}_+^n)}^2, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}_+^n}), \quad (1.14)$$

where Γ stands for the gamma function. Our approach relies on a contraction argument in the space $BC((0, \infty); \mathcal{X}_{p,q})$ which does not require (1.13) nor (1.14). For this purpose, we need to prove estimates in weak- L^p for some boundary operators linked to (1.11). In view of (1.3), these estimates need to be time-independent and thereby one cannot use time-weighted norms *ala kato* (see [28] for this type of norm), making things more difficult-to-treating. This situation leads us to derive boundary estimates in spirit of the paper [43] that dealt with the heat and Stokes operators inside a half-space (among other smooth domains Ω). So, in a certain sense, Lemma 4.3 can be seen as extensions of Yamazaki's estimates to boundary operators. Also, before obtaining Lemma 4.3, we need to prove Lemmas 4.1 and 4.2 that seems to have an interest of its own. It is worthy to comment that weak- L^p spaces are examples of shift-invariant Banach spaces of local measure for which global-in-time well-posedness theory of small solutions has been successfully developed for Navier-Stokes equations (see [31] for a nice review) and, more generally, parabolic problems with nonlinearities (and possibly other terms) defined inside the domain (see [29]).

Let us review some works concerning heat equations with singular potentials and nonlinear boundary conditions. The paper of Baras-Goldstein [8] have motivated many works concerning heat equations with singular potentials. In these results, Hardy type-inequalities play an important role in both linear and nonlinear cases. For (1.1) (potential defined inside the domain), we refer the reader to [11],[24],[42] (see also their references) for results on existence, non-existence, decay and self-similar asymptotic behavior of solutions. Versions of (1.1) with

nonlinearities $\pm u^p$ and $\pm |\nabla u|^p$ have been studied in [2],[3], [4],[12],[27],[33],[38] where the reader can find results on existence, non-existence, Fujita exponent, self-similarity, bifurcations, and blow-up. Linear and nonlinear elliptic versions of (1.1) are also often considered in the literature (see e.g. [12],[15],[16],[17],[18],[19],[40]); as well as the parabolic case, the key tool used in the analysis is Hardy type inequalities, except by [18] and [19]. In these last two references, the authors employed a contraction argument in a sum of weighted spaces and in a space based on Fourier transform, respectively. In a bounded domain Ω and half-space \mathbb{R}_+^n , the nonlinear problem (1.5)-(1.7) with $V \equiv 0$ has been studied by several authors over the past two decades; see, e.g., [5],[6],[23],[26],[37],[39] and their references. In these works, the reader can find many types of existence and asymptotic behavior results in the framework of L^p -spaces. For $V \in L^\infty(\partial\Omega)$ and Ω a bounded smooth domain, results on well-posedness and attractors can be found in [7]. The authors of [13] considered (1.5)-(1.7) with $h(u) \equiv 0$ (linear case) and showed L^p -estimates of solutions, still for $V \in L^\infty(\partial\Omega)$ (see also [14] for the elliptic case). In [25], the authors studied the linear case of (1.5)-(1.7) in a half-space and considered the singular critical potential $V(x') = \frac{\lambda}{|x'|}$. For compactly supported data $u_0 \in C_0(\mathbb{R}_+^n)$, they obtained a threshold value for existence of positive solutions by using the Kato inequality (1.14).

In this paragraph, we summarize the novelties of the present paper in comparison with the previous ones. Our results provide a global-in-time well-posedness theory for (1.5)-(1.7) in a framework that is larger than L^p -spaces and seems to be new in the study of parabolic problems with nonlinear boundary conditions. Also, among others, it allows us to consider critical potentials on the boundary with infinite many singularities which are not covered by previous results. As pointed out above, a remarkable difference is that the approach employed here does not use Hardy nor Kato inequalities, being based on boundary estimates on weak- L^p spaces. Since the smallness condition on u_0 is with respect to the weak norm of such spaces, some initial data with large L^p and H^s -norms can be considered. Results on self-similarity and axial-symmetry are naturally obtained due to the choice of the space indexes and to the symmetry features of the linear operators arising in the integral formulation (1.11).

The plan of this paper is the following. In the next section we summarize some basic definitions and properties on Lorentz spaces. In section 3 we define suitable time-functional spaces and state our results, which are proved in section 4.

2 Preliminaries

In this section we fix some notations and summarize basic properties about Lorentz spaces that will be used throughout the paper. For further details, we refer the reader to [9],[10].

For a point $x \in \overline{\mathbb{R}_+^n}$, we write $x = (x', x_n)$ where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \geq 0$. The Lebesgue measure in a measurable $\Omega \subset \mathbb{R}^n$ will be denoted by either $|\cdot|$ or dx . In the case $\Omega = \mathbb{R}_+^n$, one can express $dx = dx' dx_n$ where dx' stands for Lebesgue measure on $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1}$. Given a subset $\Omega \subset \mathbb{R}^n$, the distribution function and rearrangement of a

measurable function $f : \Omega \rightarrow \mathbb{R}$ is defined respectively by

$$\lambda_f(s) = |\{x \in \Omega : |f(x)| > s\}| \quad \text{and} \quad f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}, t > 0.$$

The *Lorentz space* $L^{(p,r)} = L^{(p,r)}(\Omega) = L^{(p,r)}(\Omega, |\cdot|)$ consists of all measurable functions f in Ω for which

$$\|f\|_{L^{(p,r)}(\Omega)}^* = \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}} [f^*(t)] \right)^r \frac{dt}{t} \right]^{\frac{1}{r}} < \infty, & 0 < p < \infty, 1 \leq r < \infty \\ \sup_{t>0} t^{\frac{1}{p}} [f^*(t)] < \infty, & 0 < p < \infty, r = \infty. \end{cases} \quad (2.1)$$

We have that $L^p(\Omega) = L^{(p,p)}(\Omega)$ and $L^{(p,\infty)}$ is also called weak- L^p or Marcinkiewicz space. The quantity (2.1) is not a norm in $L^{(p,r)}$, however it is a complete quasi-norm. Considering

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds,$$

we can endow $L^{(p,r)}$ with the quantity $\|\cdot\|_{L^{(p,r)}}$ obtained from (2.1) with f^{**} in place of f^* . For $1 < p \leq \infty$, we have that $\|\cdot\|_{(p,r)}^* \leq \|\cdot\|_{(p,r)} \leq \frac{p}{p-1} \|\cdot\|_{(p,r)}^*$ which implies that $\|\cdot\|_{(p,r)}^*$ and $\|\cdot\|_{(p,r)}$ induce the same topology on $L^{(p,r)}$. Moreover, the pair $(L^{(p,r)}, \|\cdot\|_{L^{(p,r)}})$ is a Banach space. From now on, for $1 < p \leq \infty$ we consider $L^{(p,r)}$ endowed with $\|\cdot\|_{L^{(p,r)}}$, except when explicitly mentioned.

If, for $\lambda > 0$, $\lambda\Omega = \{\lambda x : x \in \Omega\}$ is the *dilation* of the domain Ω , then

$$\|f(\lambda x)\|_{L^{(p,r)}(\Omega)} = \lambda^{-\frac{n}{p}} \|f(x)\|_{L^{(p,r)}(\Omega)}, \quad (2.2)$$

provided that Ω is invariant by dilations, i.e., $\Omega = \lambda\Omega$.

For $1 \leq q_1 \leq p \leq q_2 \leq \infty$ with $1 < p \leq \infty$, the continuous inclusions hold true

$$L^{(p,1)} \subset L^{(p,q_1)} \subset L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)}.$$

The dual space of $L^{(p,r)}$ is $L^{(p',r')}$ for $1 \leq p, r < \infty$. In particular, the dual of $L^{(p,1)}$ is $L^{(p',\infty)}$ for $1 \leq p < \infty$.

Hölder's inequality works well in Lorentz spaces (see [34]). Precisely, if $1 < p_1, p_2, p_3 < \infty$ and $1 \leq r_1, r_2, r_3 \leq \infty$ with $1/p_3 = 1/p_1 + 1/p_2$ and $1/r_3 \leq 1/r_1 + 1/r_2$, then

$$\|fg\|_{L^{(p_3,r_3)}} \leq C \|f\|_{L^{(p_1,r_1)}} \|g\|_{L^{(p_2,r_2)}}, \quad (2.3)$$

where $C > 0$ is a constant independent of f, g .

Finally we recall some interpolation property of Lorentz spaces. For $0 < p_1 < p_2 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \leq r_1, r_2, r \leq \infty$, we have that (see [10, Theorems 5.3.1, 5.3.2])

$$(L^{(p_1,r_1)}, L^{(p_2,r_2)})_{\theta,r} = L^{(p,r)}, \quad (2.4)$$

where $(X, Y)_{\theta, r}$ stands for the real interpolation space between X and Y constructed via the $K_{\theta, q}$ -method. It is well known that $(\cdot, \cdot)_{\theta, r}$ is an exact interpolation functor of exponent θ on the categories of quasi-normed and normed spaces. When $0 < p_1 \leq 1$, the property (2.4) should be considered with $L^{(p_1, r_1)}$ endowed with the complete quasi-norm $\|\cdot\|_{L^{(p_1, r_1)}}^*$ instead of $\|\cdot\|_{L^{(p_1, r_1)}}$.

3 Functional setting and results

Before starting our results, we define suitable function spaces where (1.11) will be handled. If the potential V is a homogeneous function of degree -1 , that is, $V(y) = \lambda V(\lambda y)$ for all $y \in \partial\mathbb{R}_+^n$, then $u_\lambda(x, t) = \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t)$ verifies (1.5)-(1.6), for each fixed $\lambda > 0$, provided that $u(x, t)$ is also a solution. It follows that (1.5)-(1.6) has the following scaling

$$u(x, t) \rightarrow u_\lambda(x, t) = \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t), \quad \lambda > 0. \quad (3.1)$$

Making $t \rightarrow 0^+$ in (3.1), one obtains

$$u_0(x) \rightarrow u_{0, \lambda}(x, 0) = \lambda^{\frac{1}{\rho-1}} u_0(\lambda x), \quad (3.2)$$

which gives a scaling for the initial data.

Since the potential V and initial data u_0 are singular, we need to treat (1.11) in a suitable space of functions without any positive regularity conditions and time decaying. For that matter, let \mathcal{A} be the set of measurable functions $f : \overline{\mathbb{R}_+^n} \rightarrow \mathbb{R}$ such that $f|_{\mathbb{R}_+^n}$ and $f|_{\partial\mathbb{R}_+^n}$ are measurable with respect to Lebesgue σ -algebra on \mathbb{R}_+^n and $\mathbb{R}^{n-1} = \partial\mathbb{R}_+^n$, respectively. Consider the equivalence relation in \mathcal{A} : $f \sim g$ if and only if $f = g$ a.e. in \mathbb{R}_+^n and $f|_{\partial\mathbb{R}_+^n} = g|_{\partial\mathbb{R}_+^n}$ a.e. in $\partial\mathbb{R}_+^n$. Given $1 \leq p, q < \infty$, we set $\mathcal{X}_{p, q}$ as the space of all $f \in \mathcal{A} / \sim$ such that

$$\|f\|_{\mathcal{X}_{p, q}} = \|f\|_{L^{(p, \infty)}(\mathbb{R}_+^n)} + \|f|_{\partial\mathbb{R}_+^n}\|_{L^{(q, \infty)}(\partial\mathbb{R}_+^n)} < \infty.$$

The pair $(\mathcal{X}_{p, q}, \|\cdot\|_{\mathcal{X}_{p, q}})$ is a Banach space and can be isometrically identified with $L^{(p, \infty)}(\mathbb{R}_+^n) \times L^{(q, \infty)}(\partial\mathbb{R}_+^n)$. For $p = n(\rho - 1)$ and $q = (n - 1)(\rho - 1)$, we have from (2.2) that

$$\|\lambda^{\frac{1}{\rho-1}} f(\lambda x)\|_{\mathcal{X}_{p, q}} = \lambda^{\frac{1}{\rho-1}} \lambda^{-\frac{n}{n(\rho-1)}} \|f\|_{L^{(p, \infty)}(\mathbb{R}_+^n)} + \lambda^{\frac{1}{\rho-1}} \lambda^{-\frac{n-1}{(n-1)(\rho-1)}} \|f|_{\partial\mathbb{R}_+^n}\|_{L^{(q, \infty)}(\partial\mathbb{R}_+^n)} = \|f\|_{\mathcal{X}_{p, q}},$$

and then $\mathcal{X}_{p, q}$ is invariant by scaling (3.2).

We shall look for solutions in the Banach space $E = BC((0, \infty); \mathcal{X}_{p, q})$ endowed with the norm

$$\|u\|_E = \sup_{t>0} \|u(\cdot, t)\|_{\mathcal{X}_{p, q}}, \quad (3.3)$$

which is invariant by scaling (3.1).

3.1 Existence and self-similarity

In what follows, we state our well-posedness result.

Theorem 3.1. *Let $n \geq 3$, $\rho > 1$ with $\frac{\rho}{\rho-1} < n-1$, $p = n(\rho-1)$ and $q = (n-1)(\rho-1)$. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ verify (1.8) and $h(0) = 0$. Suppose that $V \in L^{(n-1, \infty)}(\partial\mathbb{R}_+^n)$ and $u_0 \in L^{(p, \infty)}(\mathbb{R}_+^n)$.*

- (A) *(Existence and uniqueness) There exist $\varepsilon, \delta_1, \delta_2 > 0$ such that if $\|V\|_{L^{(n-1, \infty)}(\partial\mathbb{R}_+^n)} < \frac{1}{\delta_1}$ and $\|u_0\|_{L^{(p, \infty)}(\mathbb{R}_+^n)} \leq \frac{\varepsilon}{\delta_2}$ then the integral equation (1.11) has a unique solution $u \in BC((0, \infty); \mathcal{X}_{p,q})$ satisfying $\sup_{t>0} \|u(\cdot, t)\|_{\mathcal{X}_{p,q}} \leq \frac{2\varepsilon}{1-\gamma}$ where $\gamma = \delta_1 \|V\|_{L^{(n-1, \infty)}(\partial\mathbb{R}_+^n)}$. Moreover, $u(\cdot, t) \rightharpoonup u_0$ in $\mathcal{S}'(\mathbb{R}_+^n)$ as $t \rightarrow 0^+$.*
- (B) *(Continuous dependence) The solution obtained in item (A) depends continuously on the initial data u_0 and potential V .*

Remark 3.2.

- (A) *The integral solution $u(\cdot, t) \in \mathcal{X}_{p,q}$, for each $t > 0$, even requiring only $u_0 \in L^{(p, \infty)}(\mathbb{R}_+^n)$. It is a kind of “parabolic regularizing effect” in the sense that solutions verifies a property for $t > 0$ that is not necessarily verified by initial data. Here, it comes essentially from the fact that $u_1(x, t) = \int_{\mathbb{R}_+^n} G(x, y, t) u_0(y) dy$ has a trace well-defined on $\partial\mathbb{R}_+^n$ and $u_1|_{\partial\mathbb{R}_+^n} \in L^{(q, \infty)}(\partial\mathbb{R}_+^n)$, for each $t > 0$, even if the data u_0 does not have a trace. Moreover, the estimate (4.4) provides a control on the trace just using the norm of u_0 in $L^{(p, \infty)}(\mathbb{R}_+^n)$.*
- (B) *The time-continuity of the solution $u(\cdot, t) \in \mathcal{X}_{p,q}$ at $t > 0$ comes naturally from the uniform continuity of the kernel $G(x, y, t)$ (1.12) on $\overline{\mathbb{R}_+^n} \times \overline{\mathbb{R}_+^n} \times [\delta, \infty)$, for each fixed $\delta > 0$.*
- (C) *A standard argument shows that solutions of (1.11) obtained in Theorem 3.1 verifies (1.5)-(1.7) in the sense of distributions.*

Since the spaces in which we look for solutions are invariant by scaling (3.1), it is natural to ask about existence of self-similar solutions. This issue is considered in the next theorem.

Theorem 3.3. *Let u be the mild solution obtained in Theorem 3.1 corresponding to the triple $(u_0, V, h(\cdot))$. If u_0, V and $h(\cdot)$ are homogeneous functions of degree $\frac{-1}{\rho-1}$, -1 and ρ , respectively, then u is a self-similar solution, i.e.,*

$$u(x, t) \equiv u_\lambda(x, t) := \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t), \text{ for all } \lambda > 0.$$

3.2 Symmetries and positivity

In this subsection, we are concerned with symmetry and positivity of solutions. It is easy to see that the fundamental solution (1.12) is positive and invariant by the set \mathcal{O}_{x_n} of all rotations around the axis $\overrightarrow{Ox_n}$. Because of that, it is natural to wonder whether solutions obtained in Theorem 3.1 present positivity and symmetry properties, under certain conditions on the data and potential.

For that matter, let \mathcal{A} be a subset of \mathcal{O}_{x_n} . We recall that a function f is symmetric under the action of \mathcal{A} when $f(x) = f(T(x))$ for any $T \in \mathcal{A}$. If $f(x) = -f(T(x))$ for all $T \in \mathcal{A}$, then f is said to be antisymmetric under \mathcal{A} .

Theorem 3.4. *Under the hypotheses of Theorem 3.1. Let $\mathcal{U} \subset \mathbb{R}_+^n$ be a positive-measure set and \mathcal{A} a subset of \mathcal{O}_{x_n} .*

- (A) *Let $h(a) \geq 0$ (resp. ≤ 0) when $a \geq 0$ (resp. ≤ 0). If $u_0 \geq 0$ (resp. ≤ 0) a.e. in \mathbb{R}_+^n , $u_0 > 0$ (resp. < 0) in \mathcal{U} , and $V \geq 0$ in $\partial\mathbb{R}_+^n$, then u is positive (resp. negative) in $\mathbb{R}_+^n \times (0, \infty)$.*
- (B) *Let $h(a) = -h(-a)$, for all $a \in \mathbb{R}$, and let V be symmetric under the action of $\mathcal{A}|_{\partial\mathbb{R}_+^n}$. For all $t > 0$, the solution $u(\cdot, t)$ is symmetric (resp. antisymmetric), when u_0 is symmetric (resp. antisymmetric) under \mathcal{A} .*

Remark 3.5. (Special cases of symmetries) *Let $h(a) = -h(-a)$, for all $a \in \mathbb{R}$.*

- (i) *Consider $\mathcal{A} = \mathcal{O}_{x_n}$ and let V be radially symmetric on \mathbb{R}^{n-1} . We obtain from item (B) that if u_0 is invariant under rotations around the axis $\overrightarrow{Ox_n}$ then $u(\cdot, t)$ does so, for all $t > 0$.*
- (ii) *Let $\mathcal{A} = \{T_{x_n}\}$ where T_{x_n} is the reflection with respect to $\overrightarrow{Ox_n}$, i.e., $T_{x_n}((x', x_n)) = (-x', x_n)$ for all $x = (x', x_n)$ and $x_n \geq 0$. A function f is said to be $\overrightarrow{Ox_n}$ -even (resp. $\overrightarrow{Ox_n}$ -odd) when f is symmetric (resp. antisymmetric) under $\{T_{x_n}\}$. If $V(x)$ is an even function then the solution $u(\cdot, t)$ is $\overrightarrow{Ox_n}$ -even (resp. $\overrightarrow{Ox_n}$ -odd), for all $t > 0$, provided that u_0 is $\overrightarrow{Ox_n}$ -even (resp. $\overrightarrow{Ox_n}$ -odd).*

Remark 3.6. *Combining Theorems 3.3 and 3.4, we can obtain solutions that are both self-similar and invariant by rotations around $\overrightarrow{Ox_n}$. For instance, in the case $h(a) = \pm |a|^{\rho-1} a$, just take*

$$V(x') = \kappa |x'|^{-1} \text{ and } u_0(x) = \theta \left(\frac{x_n}{|x|} \right) |x|^{-\frac{1}{\rho-1}}$$

where $\theta(z) \in BC(\mathbb{R})$ and κ is a constant.

4 Proofs

This section is devoted to the proofs of the results. We start by estimating in Lorentz spaces some linear operators appearing in the integral formulation (1.11)

4.1 Linear Estimates

Let $f|_0 = f(x', 0)$ stand for the restriction of f to $\partial\mathbb{R}_+^n = \mathbb{R}^{n-1}$. We also denote by $\{E(t)\}_{t \geq 0}$ the heat semigroup in the half-space, namely

$$E(t)f(x) = \int_{\mathbb{R}_+^n} G(x, y, t) f(y) dy \quad (4.1)$$

where $G(x, y, t)$ is the fundamental solution given in (1.12). For $\delta > 0$ and $1 \leq q_1 \leq q_2 \leq \infty$, let us recall the well-known L^q -estimate for the heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ on \mathbb{R}^n :

$$\|(-\Delta_x)^{\frac{\delta}{2}} e^{t\Delta} f\|_{L^{q_2}(\mathbb{R}^n)} \leq C t^{-\frac{1}{2}(\frac{n}{q_1} - \frac{n}{q_2}) - \frac{\delta}{2}} \|f\|_{L^{q_1}(\mathbb{R}^n)}, \quad (4.2)$$

where $C > 0$ is a constant independent of f and t , and $(-\Delta_x)^{\frac{\delta}{2}}$ stands for the Riesz potential. For $0 < \delta < n$ and $1 \leq q_1 < q_2 < \infty$ such that $\frac{1}{\delta} < q_1 < \frac{n}{\delta}$ and $\frac{n-1}{q_2} = \frac{n}{q_1} - \delta$, we have the Sobolev trace-type inequality in L^p (see [1, Theorem 2])

$$\|f|_0\|_{L^{q_2}(\partial\mathbb{R}_+^n)} \leq C \left\| (-\Delta_x)^{\frac{\delta}{2}} f \right\|_{L^{q_1}(\mathbb{R}^n)}. \quad (4.3)$$

The next lemma provide a boundary estimate for (4.1) in the setting of Lorentz spaces.

Lemma 4.1. *Let $1 < d_1 < d_2 < \infty$ and $1 \leq r \leq \infty$. Then there exists a constant $C > 0$ such that*

$$\|[E(t)f]|_0\|_{L^{(d_2, r)}(\partial\mathbb{R}_+^n)} \leq C t^{-\left(\frac{n}{2d_1} - \frac{n-1}{2d_2}\right)} \|f\|_{L^{(d_1, r)}(\mathbb{R}_+^n)}, \quad (4.4)$$

for all $f \in L^{(d_1, r)}(\mathbb{R}_+^n)$ and $t > 0$.

Proof. Consider the extension from \mathbb{R}_+^n to \mathbb{R}^n

$$\tilde{f}(x) = \begin{cases} f(x', x_n), & x_n > 0 \\ f(x', -x_n), & x_n \leq 0. \end{cases}$$

Now notice that

$$E(t)f(x) = e^{t\Delta} \tilde{f}(x) = (g(\cdot, t) * \tilde{f})(x)$$

where

$$g(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \quad (4.5)$$

is the heat kernel on the whole space \mathbb{R}^n . Therefore, $e^{t\Delta}\tilde{f}(x)$ is an extension from \mathbb{R}_+^n to \mathbb{R}^n of $E(t)f(x)$ and

$$[E(t)f]|_0 = [e^{t\Delta}\tilde{f}]|_0. \quad (4.6)$$

Let $0 < \delta < 1$, $\frac{1}{\delta} < l < \frac{n}{\delta}$ and $d_2 > r$ be such that $\frac{n-1}{d_2} = \frac{n}{l} - \delta$. It follows from (4.6) and (4.3) that

$$\begin{aligned} \|[E(t)f]|_0\|_{L^{d_2}(\partial\mathbb{R}_+^n)} &\leq C\|(-\Delta_x)^{\frac{\delta}{2}}e^{t\Delta}\tilde{f}\|_{L^l(\mathbb{R}^n)} \\ &\leq Ct^{-\frac{1}{2}\left(\frac{n}{d_1}-\frac{n}{l}\right)-\frac{\delta}{2}}\|\tilde{f}\|_{L^{d_1}(\mathbb{R}^n)} \\ &\leq Ct^{-\frac{1}{2}\left(\frac{n}{d_1}-\frac{n-1}{d_2}\right)}\|f\|_{L^{d_1}(\mathbb{R}_+^n)}. \end{aligned}$$

Now a real interpolation argument leads us

$$\|[E(t)f]|_0\|_{L^{(d_2,r)}(\partial\mathbb{R}_+^n)} \leq Ct^{-\left(\frac{n}{2d_1}-\frac{n-1}{2d_2}\right)}\|f\|_{L^{(d_1,r)}(\mathbb{R}_+^n)}.$$

■

Let us define the integral operators

$$\mathcal{G}_1(\varphi)(x, t) = \int_{\partial\mathbb{R}_+^n} G(x, y', t)\varphi(y')dy' \quad \text{and} \quad \mathcal{G}_2(\varphi)(y', t) = \int_{\mathbb{R}_+^n} G(x, y', t)\varphi(x)dx$$

where $G(x, y, t)$ is defined in (1.12). Notice that the functions $G(x, y', t)$ and $\mathcal{G}_1(\varphi)(x, t)$ are also well-defined for $x = (x', x_n) \in \mathbb{R}^n$. Recall the pointwise estimate for the heat kernel (4.5) on \mathbb{R}^n

$$|(-\Delta_x)^{\frac{\delta}{2}}g(x, t)| \leq \frac{C_\delta}{(t + |x|^2)^{\frac{n}{2} + \frac{\delta}{2}}} \quad (\delta \geq 0), \quad (4.7)$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

Lemma 4.2. *Let $1 < d_1 < d_2 < \infty$ and $1 \leq r \leq \infty$. Then, there exists $C > 0$ such that*

$$\|\mathcal{G}_1(\psi)(x', 0, t)\|_{L^{(d_2,r)}(\partial\mathbb{R}_+^n, dx')} \leq Ct^{-\left(\frac{n-1}{2d_1}-\frac{n-1}{2d_2}+\frac{1}{2}\right)}\|\psi\|_{L^{(d_1,r)}(\partial\mathbb{R}_+^n, dx')} \quad (4.8)$$

$$\|\mathcal{G}_1(\psi)(x, t)\|_{L^{(d_2,r)}(\mathbb{R}_+^n, dx)} \leq Ct^{-\left(\frac{n-1}{2d_1}-\frac{n}{2d_2}+\frac{1}{2}\right)}\|\psi\|_{L^{(d_1,r)}(\partial\mathbb{R}_+^n, dx')} \quad (4.9)$$

for all $\psi \in L^{(d_1,r)}(\partial\mathbb{R}_+^n)$.

Proof. Let $0 < \delta < 1$, $\frac{1}{\delta} < l < \frac{n}{\delta}$ and $\frac{n-1}{d_2} = \frac{n}{l} - \delta$. Firstly, notice that the trace-type inequality (4.3) yields

$$\|\mathcal{G}_1(\varphi)(x', 0, t)\|_{L^{d_2}(\partial\mathbb{R}_+^n)} \leq C\|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x', x_n, t)\|_{L^l(\mathbb{R}^n)}. \quad (4.10)$$

Next we employ Minkowski's inequality for integrals and the pointwise estimate (4.7) to obtain

$$\begin{aligned}
\|(-\Delta_x)^{\frac{\delta}{2}} \mathcal{G}_1(\varphi)(x', x_n, t)\|_{L^l(\mathbb{R}, dx_n)} &= \left(\int_{-\infty}^{\infty} |(-\Delta_x)^{\frac{\delta}{2}} \mathcal{G}_1(\varphi)(x', x_n, t)|^l dx_n \right)^{\frac{1}{l}} \\
&\leq C \int_{\partial \mathbb{R}_+^n} \left(\int_{-\infty}^{\infty} \frac{|\varphi(y')|^l}{(t + |x' - y'|^2 + x_n^2)^{\frac{n}{2} + \frac{\delta l}{2}}} dx_n \right)^{\frac{1}{l}} dy' \\
&= 2C \int_{\partial \mathbb{R}_+^n} |\varphi(y')| \left(\int_0^{\infty} \frac{dx_n}{(t + |x' - y'|^2 + x_n^2)^{\frac{n}{2} + \frac{\delta l}{2}}} \right)^{\frac{1}{l}} dy' \\
&= 2C \int_{\partial \mathbb{R}_+^n} |\varphi(y')| (t + |x' - y'|^2)^{-\frac{n}{2} - \frac{\delta}{2} + \frac{1}{2l}} dy' \quad (4.11) \\
&\leq Ct^{-(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l})\theta} \int_{\partial \mathbb{R}_+^n} |x' - y'|^{-2(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l})(1-\theta)} |\varphi(y')| dy' \quad (4.12)
\end{aligned}$$

where (4.12) is obtained from (4.11) by using that $(a+b)^{-k} \leq a^{-k\theta} b^{-k(1-\theta)}$ when $0 < \theta < 1$ and $\kappa \geq 0$. Let $d_1 < l$ and $\gamma = (n-1)(\frac{1}{d_1} - \frac{1}{l})$. Let $0 < \theta < 1$ be such that $(n-1) - \gamma = (n + \delta - \frac{1}{l})(1 - \theta)$. It follows that $\frac{1}{l} = \frac{1}{d_1} - \frac{\gamma}{n-1} > 0$, and Sobolev embedding theorem gives us

$$\left\| \int_{\partial \mathbb{R}_+^n} \frac{1}{|x' - y'|^{(n-1)-\gamma}} |\varphi(y')| dy' \right\|_{L^l(\partial \mathbb{R}_+^n)} \leq C \|\varphi\|_{L^{d_1}(\partial \mathbb{R}_+^n)}. \quad (4.13)$$

Fubini's theorem, (4.12) and (4.13) imply that

$$\begin{aligned}
\|(-\Delta_x)^{\frac{\delta}{2}} \mathcal{G}_1(\varphi)(x', x_n, t)\|_{L^l(\mathbb{R}^n)} &= \left\| \|(-\Delta_x)^{\frac{\delta}{2}} \mathcal{G}_1(\varphi)(x', x_n, t)\|_{L^l(\mathbb{R}, dx_n)} \right\|_{L^l(\mathbb{R}^{n-1}, dx')} \\
&\leq Ct^{-(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l})\theta} \left\| \int_{\partial \mathbb{R}_+^n} \frac{1}{|x' - y'|^{(n-1)-\gamma}} |\varphi(y')| dy' \right\|_{L^l(\partial \mathbb{R}_+^n, dx')} \\
&\leq Ct^{-(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l})\theta} \|\varphi\|_{L^{d_1}(\partial \mathbb{R}_+^n)}. \quad (4.14)
\end{aligned}$$

It follows from (4.14) and (4.10) that

$$\|\mathcal{G}_1(\varphi)(x', 0, t)\|_{L^{d_2}(\partial \mathbb{R}_+^n)} \leq Ct^{-(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l})\theta} \|\varphi\|_{L^{d_1}(\partial \mathbb{R}_+^n)} = Ct^{\frac{n-1}{2d_2} - \frac{n-1}{2d_1} - \frac{1}{2}} \|\varphi\|_{L^{d_1}(\partial \mathbb{R}_+^n)}, \quad (4.15)$$

because of the equality

$$\begin{aligned}
-\left(\frac{n}{2} + \frac{\delta}{2} - \frac{1}{2l}\right)\theta &= \frac{n-1}{2} - \frac{\gamma}{2} - \frac{1}{2}\left(n + \delta - \frac{1}{l}\right) \\
&= \frac{n-1}{2d_2} - \frac{n-1}{2d_1} - \frac{1}{2}.
\end{aligned}$$

Now the estimate (4.8) follows from (4.15) and real interpolation. The proof of (4.9) is similar and is left to the reader. ■

In the next lemma we obtain refined boundary estimates on the Lorentz space $L^{(d,1)}$ that is the pre-dual one of $L^{(d',\infty)}$. These can be seen as extensions of Yamazaki's estimates (see [43]) to the operators \mathcal{G}_1 and \mathcal{G}_2 .

Lemma 4.3. *Let $1 < d_1 < d_2 < \infty$, there exists a constant $C > 0$ such that*

$$\int_0^\infty t^{\left(\frac{n-1}{2d_1} - \frac{n-1}{2d_2}\right) - \frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot, 0, t)\|_{L^{(d_2,1)}(\partial\mathbb{R}_+^n)} dt \leq C \|\psi\|_{L^{(d_1,1)}(\partial\mathbb{R}_+^n)} \quad (4.16)$$

$$\int_0^\infty t^{\left(\frac{n-1}{2d_1} - \frac{n}{2d_2}\right) - \frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot, t)\|_{L^{(d_2,1)}(\mathbb{R}_+^n)} dt \leq C \|\psi\|_{L^{(d_1,1)}(\partial\mathbb{R}_+^n)} \quad (4.17)$$

$$\int_0^\infty t^{\left(\frac{n}{2d_1} - \frac{n-1}{2d_2}\right) - 1} \|\mathcal{G}_2(\varphi)(\cdot, t)\|_{L^{(d_2,1)}(\partial\mathbb{R}_+^n)} dt \leq C \|\varphi\|_{L^{(d_1,1)}(\mathbb{R}_+^n)}, \quad (4.18)$$

for all $\psi \in L^{(d_1,1)}(\partial\mathbb{R}_+^n)$ and $\varphi \in L^{(d_1,1)}(\mathbb{R}_+^n)$.

Proof. We start with (4.18). Let $1 < p_1 < d_1 < p_2 < d_2$ be such that $\frac{1}{d_1} - \frac{1}{p_1} < -\frac{2}{n}$ and $\frac{1}{d_1} - \frac{1}{p_2} < 2$. Noting that $\mathcal{G}_2(\varphi)(y', t) = [E(t)\varphi](y', 0)$, Lemma 4.1 yields

$$\|\mathcal{G}_2(\varphi)(\cdot, t)\|_{L^{(d_2,1)}(\partial\mathbb{R}_+^n)} \leq C t^{-\left(\frac{n}{2p_k} - \frac{n-1}{2d_2}\right)} \|\varphi\|_{L^{(p_k,1)}(\mathbb{R}_+^n)}, \text{ for } k = 1, 2. \quad (4.19)$$

For $\varphi \in L^{(p_1,\infty)}(\mathbb{R}_+^n) \cap L^{(p_2,\infty)}(\mathbb{R}_+^n)$, we define the following sub-linear operator

$$\mathcal{F}(\varphi)(t) = t^{\frac{n}{2d_1} - \frac{n-1}{2d_2} - 1} \|\mathcal{G}_2(\varphi)(\cdot, t)\|_{L^{(d_2,1)}(\partial\mathbb{R}_+^n)}.$$

Since $1 < p_k < d_2$, it follows from (4.19) that

$$\mathcal{F}(\varphi)(t) \leq C t^{\left(\frac{n}{2d_1} - \frac{n}{2p_k}\right) - 1} \|\varphi\|_{L^{(p_k,1)}(\mathbb{R}_+^n)}.$$

Let $\frac{1}{s_k} = 1 - \left(\frac{n}{2d_1} - \frac{n}{2p_k}\right)$ and take $0 < \theta < 1$ such that $\frac{1}{d_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then $\frac{1-\theta}{s_1} + \frac{\theta}{s_2} = 1$ with $0 < s_1 < 1 < s_2$. Therefore,

$$\begin{aligned} \|\mathcal{F}(\varphi)(t)\|_{L^{(s_k,\infty)}(0,\infty)} &\leq C \|t^{-1/s_k}\|_{L^{(s_k,\infty)}(0,\infty)}^* \|\varphi\|_{L^{(p_k,1)}(\mathbb{R}_+^n)} \\ &\leq C \|\varphi\|_{L^{(p_k,1)}(\mathbb{R}_+^n)}, \end{aligned}$$

and so $\mathcal{F} : L^{(p_k,1)}(\mathbb{R}_+^n) \rightarrow L^{(s_k,\infty)}(0,\infty)$ is a bounded sublinear operator, for $k = 1, 2$. Taking

$$m_k = \|\mathcal{F}(\varphi)\|_{L^{(p_k,1)}(\mathbb{R}_+^n) \rightarrow L^{(s_k,\infty)}(0,\infty)},$$

and recalling the interpolation properties

$$L^{(d_1,1)} = (L^{(p_1,1)}, L^{(p_2,1)})_{\theta,1} \text{ and } L^1 = (L^{(s_1,\infty)}, L^{(s_2,\infty)})_{\theta,1},$$

we obtain

$$\|\mathcal{F}(\varphi)\|_{L^1(0,\infty)} \leq C m_1^{1-\theta} m_2^\theta \|\varphi\|_{L^{(d_1,1)}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{L^{(d_1,1)}(\mathbb{R}_+^n)},$$

which is exactly (4.18).

In order to show (4.16), now we define

$$\mathcal{F}(\psi)(t) = t^{\frac{n-1}{2d_1} - \frac{n-1}{2d_2} - \frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot, 0, t)\|_{L^{(d_2,1)}(\partial\mathbb{R}_+^n)}$$

and obtain by means of (4.8) that

$$\mathcal{F}(\psi)(t) \leq C t^{\frac{n-1}{2d_1} - \frac{n-1}{2p_k} - 1} \|\psi\|_{L^{(p_k,1)}(\partial\mathbb{R}_+^n)}.$$

Let $\frac{1}{s_k} = 1 - \left(\frac{n-1}{2d_1} - \frac{n-1}{2p_k}\right)$ and $0 < \theta < 1$ be such that $\frac{1}{d_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Then $\frac{1-\theta}{s_1} + \frac{\theta}{s_2} = 1$, and one can obtain (4.16) by proceeding similarly to proof of (4.18). The proof of (4.17) follows analogously by considering

$$\mathcal{F}(\psi)(t) = t^{\frac{n-1}{2d_1} - \frac{n}{2d_2} - \frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot, \cdot, t)\|_{L^{(d_2,1)}(\mathbb{R}_+^n)}$$

and using (4.9) instead of (4.8). The details are left to the reader. ■

4.2 Nonlinear estimates

This section is devoted to estimate the operators

$$\mathcal{N}(u)(x, t) = \int_0^t \int_{\partial\mathbb{R}_+^n} G(x, y', t-s) h(u(y', s)) dy' ds \quad (4.20)$$

$$\mathcal{T}(u)(x, t) = \int_0^t \int_{\partial\mathbb{R}_+^n} G(x, y', t-s) V(y') u(y', s) dy' ds. \quad (4.21)$$

For that matter, we define (for each fixed $t > 0$)

$$k_t(x, y', s) = \begin{cases} G(x, y', s), & \text{if } 0 < s < t \\ 0, & \text{otherwise} \end{cases}$$

and consider \mathcal{H} the boundary parabolic integral operator

$$\mathcal{H}(f)(x, t) = \int_0^\infty \int_{\partial\mathbb{R}_+^n} k_t(x, y', t-s) f(y', s) dy' ds.$$

For a suitable function φ defined in either $\Omega = \mathbb{R}_+^n$ or $\Omega = \partial\mathbb{R}_+^n$, let us denote

$$\langle \mathcal{H}(f), \varphi \rangle_\Omega = \int_\Omega \mathcal{H}(f)(x, t) \varphi(x) dx.$$

Using Tonelli's theorem, we have that

$$\begin{aligned} |\langle \mathcal{H}(f), \varphi \rangle_{\mathbb{R}_+^n}| &\leq \int_{\mathbb{R}_+^n} \int_0^\infty \left(\int_{\partial\mathbb{R}_+^n} k_t(x, y', t-s) |f(y', s)| dy' \right) ds |\varphi(x)| dx \\ &= \int_0^\infty \int_{\partial\mathbb{R}_+^n} |f(y', s)| \left(\int_{\mathbb{R}_+^n} k_t(x, y', t-s) |\varphi(x)| dx \right) dy' ds \\ &= \int_0^\infty \langle |f(\cdot, s)|, \mathcal{G}_2(|\varphi|)(\cdot, t-s) \rangle_{\partial\mathbb{R}_+^n} ds \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} |\langle \mathcal{H}f, \varphi \rangle_{\partial\mathbb{R}_+^n}| &\leq \int_0^\infty \int_{\partial\mathbb{R}_+^n} |f(y', s)| \mathcal{G}_1(|\varphi|)(y', 0, t-s) dy' ds \\ &= \int_0^\infty \langle |f(\cdot, s)|, \mathcal{G}_1(|\varphi|)(\cdot, 0, t-s) \rangle_{\partial\mathbb{R}_+^n} ds, \end{aligned} \quad (4.23)$$

because the kernel of $\mathcal{G}_1(\varphi)(\cdot, 0, t-s)$ and $\mathcal{G}_2(\varphi)(\cdot, t-s)$ is $k_t(x, y', t-s)$.

Lemma 4.4. *Let $n \geq 3$, $\frac{n-1}{n-2} < \rho < \infty$, and $q = (n-1)(\rho-1)$, $p = n(\rho-1)$. There exists a constant $C > 0$ such that*

$$\sup_{t>0} \|\mathcal{H}(f)(\cdot, t)\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} \leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}_+^n)} \quad (4.24)$$

$$\sup_{t>0} \|\mathcal{H}(f)(\cdot, t)\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)} \leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}_+^n)} \quad (4.25)$$

for all $f \in L^\infty((0, \infty); L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}_+^n))$.

Proof. Estimate (4.22) and Hölder inequality (2.3) yields

$$\begin{aligned} |\langle \mathcal{H}f, \varphi \rangle_{\mathbb{R}_+^n}| &\leq C \int_0^\infty \|f(\cdot, s)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}_+^n)} \|\mathcal{G}_2(|\varphi|)(\cdot, t-s)\|_{L^{(\frac{q}{q-\rho}, 1)}(\partial\mathbb{R}_+^n)} ds \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}_+^n)} \int_0^\infty \|\mathcal{G}_2(|\varphi|)(\cdot, t-s)\|_{L^{(\frac{q}{q-\rho}, 1)}(\partial\mathbb{R}_+^n)} ds. \end{aligned} \quad (4.26)$$

Next, notice that $(\frac{q}{\rho})' = \frac{q}{q-\rho} > p'$ and

$$\frac{1}{2} \left(\frac{n}{p'} - \frac{n-1}{(\frac{q}{\rho})'} \right) - 1 = \frac{1}{2} \left(1 - \frac{1}{\rho-1} + \frac{\rho}{\rho-1} \right) - 1 = 0.$$

In view of (4.26), we can use duality and estimate (4.18) with $d_1 = p'$ and $d_2 = \frac{q}{q-\rho}$ to obtain

$$\begin{aligned} I_1(t) &= \|\mathcal{H}(f)(\cdot, t)\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} = \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}_+^n)}=1} \left| \langle \mathcal{H}(f), \varphi \rangle_{\mathbb{R}_+^n} \right| \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)} \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}_+^n)}=1} \left(\int_0^\infty \|\mathcal{G}_2(|\varphi|)(\cdot, t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}_+^n)} ds \right) \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)} \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}_+^n)}=1} \|\varphi\|_{L^{(p',1)}(\mathbb{R}_+^n)} \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)}, \end{aligned} \tag{4.27}$$

for a.e. $t > 0$. The estimate (4.24) follows by taking the essential supremum over $(0, \infty)$ in both sides of (4.27).

Now we deal with (4.25) which is the boundary part of the norm $\|\cdot\|_{\mathcal{X}_{p,q}}$. We have that $(\frac{q}{\rho})' > q'$ and

$$\frac{1}{2} \left(\frac{n-1}{q'} - \frac{1}{(\frac{q}{\rho})'} \right) - \frac{1}{2} = \frac{n-1}{2} \left(\frac{\rho}{q} - \frac{1}{q} \right) - \frac{1}{2} = 0.$$

Proceeding similarly to proof of (4.27), but using (4.16) instead of (4.18), we obtain

$$\begin{aligned} I_2(t) &= \|\mathcal{H}(f)(\cdot, t)\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)} \\ &\leq \sup_{\|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}_+^n)}=1} \int_0^\infty \|f(\cdot, s)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)} \|\mathcal{G}_1(|\varphi|)(\cdot, 0, t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}_+^n)} ds \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)} \int_0^\infty \|\mathcal{G}_1(|\varphi|)(\cdot, 0, t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}_+^n)} ds \\ &\leq C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)} \sup_{\|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}_+^n)}=1} \|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}_+^n)} \\ &= C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}_+^n)}, \end{aligned} \tag{4.28}$$

for a.e. $t \in (0, \infty)$, which is equivalent to (4.25). ■

4.3 Proof of Theorem 3.1

Part (A): Let us write (1.11) as

$$u = E(t)u_0 + \mathcal{N}(u) + \mathcal{T}(u)$$

where the operators \mathcal{N} and \mathcal{T} are defined in (4.20) and (4.21), respectively.

Recall the heat estimate (see e.g. [41, Lemma 3.4])

$$\|E(t)u_0\|_{L^{d_2}(\mathbb{R}_+^n)} \leq Ct^{-\frac{n}{2}(\frac{1}{d_2}-\frac{1}{d_1})}\|u_0\|_{L^{d_1}(\mathbb{R}_+^n)}, \quad (4.29)$$

for $1 \leq d_1 \leq d_2 \leq \infty$. By using interpolation, (4.29) leads us to

$$\|E(t)u_0\|_{L^{(d_2,\infty)}(\mathbb{R}_+^n)} \leq Ct^{-\frac{n}{2}(\frac{1}{d_2}-\frac{1}{d_1})}\|u_0\|_{L^{(d_1,\infty)}(\mathbb{R}_+^n)}, \quad (4.30)$$

for $1 < d_1 \leq d_2 < \infty$.

We consider the Banach space $E = BC((0, \infty); \mathcal{X}_{p,q})$ endowed with the norm (3.3). Estimate (4.30) and Lemma 4.1 yield

$$\begin{aligned} \|E(t)u_0\|_E &= \sup_{t>0} \|E(t)u_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} + \sup_{t>0} \|E(t)u_0\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)} \\ &\leq C \left(\|u_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} + \|u_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} \right) \\ &= \delta_2 \|u_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} \leq \varepsilon, \end{aligned} \quad (4.31)$$

provided that $\|u_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} \leq \frac{\varepsilon}{\delta_2}$. In what follows, we estimate the operators \mathcal{T} and \mathcal{N} in order to employ a contraction argument in E . Since $\frac{\rho}{q} = \frac{1}{q} + \frac{\rho-1}{q}$, property (1.8) and Hölder's inequality (2.3) yield

$$\begin{aligned} \|h(u) - h(v)\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}_+^n)} &\leq \eta \| |u - v| (|u|^{\rho-1} + |v|^{\rho-1}) \|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}_+^n)} \\ &\leq C \|u - v\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)} (\|u\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)}^{\rho-1} + \|v\|_{L^{(q,\infty)}(\partial\mathbb{R}_+^n)}^{\rho-1}). \end{aligned} \quad (4.32)$$

Using Lemma 4.4 and (4.32), we obtain

$$\begin{aligned} \sup_{t>0} \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{X}_{p,q}} &= \sup_{t>0} \|\mathcal{H}(h(u) - h(v))\|_{\mathcal{X}_{p,q}} \\ &\leq C \sup_{t>0} \|h(u) - h(v)\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}_+^n)} \\ &\leq K \|u - v\|_E (\|u\|_E^{\rho-1} + \|v\|_E^{\rho-1}). \end{aligned}$$

Also, noting that $\frac{\rho}{q} = \frac{1}{n-1} + \frac{1}{q}$, we have that

$$\begin{aligned}
\|\mathcal{T}(u) - \mathcal{T}(v)\|_E &= \sup_{t>0} \|\mathcal{H}(V(u-v))\|_{\mathcal{X}_{p,q}} \\
&\leq C \sup_{t>0} \|V(u-v)\|_{L^{(q/\rho, \infty)}(\partial\mathbb{R}_+^n)} \\
&\leq \delta_1 \|V\|_{L^{(n-1, \infty)}(\partial\mathbb{R}_+^n)} \sup_{t>0} \|u(\cdot, t) - v(\cdot, t)\|_{L^{(q, \infty)}(\partial\mathbb{R}_+^n)} \\
&\leq \gamma \|u - v\|_E \text{ with } 0 < \gamma < 1,
\end{aligned}$$

provided that $\gamma = \delta_1 \|V\|_{L^{(n-1, \infty)}(\partial\mathbb{R}_+^n)}$. Now consider

$$\Phi(u) = E(t)u_0 + \mathcal{N}(u) + \mathcal{T}(u) \quad (4.33)$$

and the closed ball $B_\varepsilon = \{u \in \mathcal{X}_{p,q}; \|u\|_{\mathcal{X}_{p,q}} \leq \frac{2\varepsilon}{1-\gamma}\}$ where $\varepsilon > 0$ is chosen in such a way that

$$\left(\frac{2^\rho \varepsilon^{\rho-1} K}{(1-\gamma)^{\rho-1}} + \gamma \right) < 1. \quad (4.34)$$

For all $u, v \in B_\varepsilon$, we obtain that

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_E &\leq \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\mathcal{T}(u) - \mathcal{T}(v)\|_E \\
&\leq \|u - v\|_E (K \|u\|_E^{\rho-1} + K \|v\|_E^{\rho-1} + \gamma) \\
&\leq \left(K \frac{2^\rho \varepsilon^{\rho-1}}{(1-\gamma)^{\rho-1}} + \gamma \right) \|u - v\|_{\mathcal{X}_{p,q}}.
\end{aligned} \quad (4.35)$$

Noting that $\Phi(0) = E(t)u_0$, the estimates (4.31) and (4.35) yield

$$\begin{aligned}
\|\Phi(u)\|_E &\leq \|E(t)u_0\|_E + \|\Phi(u) - \Phi(0)\|_E \\
&\leq \varepsilon + (K \|u\|_E^\rho + \gamma \|u\|_E) \\
&\leq \varepsilon + \left(K \frac{2^\rho \varepsilon^\rho}{(1-\gamma)^\rho} + \gamma \frac{2\varepsilon}{1-\gamma} \right) \leq \frac{2\varepsilon}{1-\gamma},
\end{aligned}$$

for all $u \in B_\varepsilon$, because of (4.34). Then the map $\Phi : B_\varepsilon \rightarrow B_\varepsilon$ is a contraction and Banach fixed point theorem assures that there is a unique solution $u \in B_\varepsilon$ for (1.11).

The weak convergence to the initial data as $t \rightarrow 0^+$ follows from standard arguments and is left to the reader (see e.g. [21, Lemma 3.8], [29, Lemmas 3.3 and 4.8]).

Part (B): Let $u, \tilde{u} \in B_\varepsilon$ be two solutions obtained in item (A) corresponding to pairs

(V, u_0) and (\tilde{V}, \tilde{u}_0) , respectively. We have that

$$\begin{aligned}
\|u - \tilde{u}\|_E &\leq \|E(t)(u_0 - \tilde{u}_0)\|_E + \|\mathcal{N}(u) - \mathcal{N}(\tilde{u})\|_E + \|\mathcal{T}(u) - \mathcal{T}(\tilde{u})\|_E \\
&\leq \delta_2 \|u_0 - \tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} + K \|u - \tilde{u}\|_E (\|u\|_E^{\rho-1} + \|\tilde{u}\|_E^{\rho-1}) \\
&\quad + \|\mathcal{H}[(V - \tilde{V})\tilde{u} + V(u - \tilde{u})]\|_E \\
&\leq \delta_2 \|u_0 - \tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} + \|u - \tilde{u}\|_E \left(\frac{2^\rho \varepsilon^{\rho-1} K}{(1-\gamma)^{\rho-1}} \right) \\
&\quad + \delta_1 \left(\|V - \tilde{V}\|_{L^{(n-1,\infty)}} \|\tilde{u}\|_E + \|V\|_{L^{(n-1,\infty)}} \|u - \tilde{u}\|_E \right) \\
&\leq \delta_2 \|u_0 - \tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}_+^n)} + \left(\frac{2^\rho \varepsilon^{\rho-1} K}{(1-\gamma)^{\rho-1}} + \gamma \right) \|u - \tilde{u}\|_E + \frac{2\delta_1 \varepsilon}{1-\gamma} \|V - \tilde{V}\|_{L^{(n-1,\infty)}},
\end{aligned}$$

which gives the desired continuity because of (4.34). ■

4.4 Proof of Theorem 3.3

From the fixed point argument in the proof of Theorem 3.1, the solution u is the limit in the space E of the Picard sequence

$$u_1 = E(t)u_0, \quad u_{k+1} = u_1 + \mathcal{N}(u_k) + \mathcal{T}(u_k), \quad k \in \mathbb{N}, \quad (4.36)$$

where \mathcal{N} and \mathcal{T} are defined in (4.20) and (4.21), respectively. Since $u_0 \in L^{(n(\rho-1),\infty)}(\mathbb{R}_+^n)$ and $V \in L^{(n-1,\infty)}(\mathbb{R}^{n-1})$, we can take u_0 and V as homogeneous functions of degree $-\frac{1}{\rho-1}$ and -1 , respectively. Using the kernel property

$$G(x, y, t) = \lambda^n G(\lambda x, \lambda y, \lambda^2 t) \quad (4.37)$$

and homogeneity of u_0 , we have that

$$\begin{aligned}
u_1(\lambda x, \lambda^2 t) &= \int_{\mathbb{R}_+^n} G(\lambda x, y, \lambda^2 t) u_0(y) dy \\
&= \int_{\mathbb{R}_+^n} \lambda^n G(\lambda x, \lambda y, \lambda^2 t) u_0(\lambda y) dy \\
&= \lambda^{-\frac{1}{\rho-1}} \int_{\mathbb{R}_+^n} G(x, y, t) u_0(y) dy = \lambda^{-\frac{1}{\rho-1}} u_1(x, t),
\end{aligned}$$

and then u_1 is invariant by (3.1). Recalling that $f(\lambda a) = \lambda^\rho f(a)$ and assuming that

$$u_k(x, t) = u_{k,\lambda}(x, t) := \lambda^{\frac{1}{\rho-1}} u_k(\lambda x, \lambda^2 t), \quad \text{for } k \in \mathbb{N},$$

we obtain

$$\begin{aligned}
\mathcal{N}(u_k)(\lambda x, \lambda^2 t) &= \int_0^{\lambda^2 t} \int_{\partial \mathbb{R}_+^n} G(\lambda x, y', \lambda^2 t - s) h(u_k(y', s)) dy' ds \\
&= \lambda^{n-1+2} \int_0^t \int_{\partial \mathbb{R}_+^n} G(\lambda x, \lambda y', \lambda^2(t-s)) h(\lambda^{-\frac{1}{\rho-1}} \lambda^{\frac{1}{\rho-1}} u_k(\lambda y', \lambda^2 s)) dy' ds \\
&= \lambda^{n+1} \int_0^t \int_{\partial \mathbb{R}_+^n} \lambda^{-n} G(x, y', t-s) \lambda^{-\frac{\rho}{\rho-1}} h(u_k(y', s)) dy' ds \\
&= \lambda^{-\frac{1}{\rho-1}} \mathcal{N}(u_k)(x, t)
\end{aligned}$$

and, similarly, $\mathcal{T}(u_k)(\lambda x, \lambda^2 t) = \lambda^{-\frac{1}{\rho-1}} \mathcal{T}(u_k)(x, t)$. It follows that

$$\lambda^{\frac{1}{\rho-1}} u_{k+1}(\lambda x, \lambda^2 t) = u_1(x, t) + \mathcal{N}(u_k) + \mathcal{T}(u_k) = u_{k+1}(x, t)$$

and then, by induction, u_k is invariant by (3.1) for all $k \in \mathbb{N}$.

Since the norm $\|\cdot\|_E$ is invariant by (3.1) and $u_k \rightarrow u$ in E , it is easy to see that u is also invariant by (3.1), that is, it is self-similar. ■

4.5 Proof of Theorem 3.4

Part (A): Let $u_0 \geq 0$ a.e. in \mathbb{R}_+^n and $\mathcal{U} \subset \mathbb{R}_+^n$ be a positive measure set with $u_0 > 0$ in \mathcal{U} . It follows from (1.12) that

$$u_1(x, t) = \int_{\mathbb{R}_+^n} G(x, y, t) u_0(y) dy > 0 \text{ in } \overline{\mathbb{R}_+^n} \times (0, \infty).$$

By using that V is nonnegative in \mathbb{R}^{n-1} and $h(a) \geq 0$ when $a \geq 0$, one can see that $\mathcal{N}(u) + \mathcal{T}(u)$ is nonnegative in $\overline{\mathbb{R}_+^n} \times (0, \infty)$ provided that $u|_{\partial \mathbb{R}_+^n} \geq 0$. Then, an induction argument applied to the sequence (4.36) shows that $u_k > 0$ in $\overline{\mathbb{R}_+^n} \times (0, \infty)$, for all $k \in \mathbb{N}$. Since the convergence in the space E implies convergence in $L^{(p, \infty)}(\mathbb{R}_+^n)$ and in $L^{(q, \infty)}(\partial \mathbb{R}_+^n)$ for each $t > 0$, we have that (up to a subsequence) $u_k(\cdot, t) \rightarrow u(\cdot, t)$ a.e. in (\mathbb{R}_+^n, dx) and a.e. in $(\partial \mathbb{R}_+^n, dx')$ for each $t > 0$. It follows that u is a nonnegative function because pointwise convergence preserves nonnegativity. Since $u_1 > 0$, then $u = u_1 + \mathcal{N}(u) + \mathcal{T}(u) \geq u_1 + 0 > 0$ in $\overline{\mathbb{R}_+^n} \times (0, \infty)$, as desired. The proof of the statement concerning negativity is left to the reader.

Part (B): We only will prove the antisymmetric part of the statement, because the sym-

metric one is analogous. Given a $T \in \mathcal{G}$, we have

$$\begin{aligned}
u_1(T(x), t) &= \int_{\mathbb{R}_+^n} G(T(x), y, t) u_0(y) dy \\
&= \int_{\mathbb{R}_+^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|T(x)-y|^2}{4t}} + e^{-\frac{|T(x)-y^*|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}_+^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|T((x-T^{-1}(y)))|^2}{4t}} + e^{-\frac{|T(x-T^{-1}(y^*))|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}_+^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|x-T^{-1}(y)|^2}{4t}} + e^{-\frac{|x-(T^{-1}(y))^*|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}_+^n} G(x, T^{-1}(y), t) u(y) dy.
\end{aligned}$$

Making the change of variable $z = T^{-1}(y)$ and using that u_0 is antisymmetric under \mathcal{G} , we obtain

$$u_1(T(x), t) = \int_{\mathbb{R}_+^n} G(x, z, t) u_0(T(z)) dz = - \int_{\mathbb{R}_+^n} G(x, z, t) u_0(z) dz = -u_1(x, t).$$

A similar argument shows that

$$\mathcal{L}(\theta)(x, t) = \int_0^t \int_{\partial\mathbb{R}_+^n} G(x, y', t-s) \theta(y', s) dy' ds$$

is antisymmetric when $\theta(\cdot, t)|_{\partial\mathbb{R}_+^n}$ is also, for each $t > 0$. As V is symmetric and $h(a) = -h(-a)$, it follows that

$$\theta(x, t) = h(u(\cdot, t)) + V u(\cdot, t)$$

is antisymmetric whenever $u(\cdot, t)$ does so. Therefore, by means of an induction argument, one can prove that each element $u_k(\cdot, t)$ of the sequence (4.36) is antisymmetric. Recall from Part (A) that (up a subsequence) $u_k(\cdot, t) \rightarrow u(\cdot, t)$ a.e. in (\mathbb{R}_+^n, dx) and in $(\partial\mathbb{R}_+^n, dx')$, for each $t > 0$. Since this convergence preserves antisymmetry, it follows that $u(\cdot, t)$ is also antisymmetric, for each $t > 0$. ■

References

- [1] D. R. Adams, Traces of potentials arising from translation invariant operators, Ann. Scuola Norm. Sup. Pisa (3) **25** (1971), 203–217.
- [2] B. Abdellaoui, I. Peral, A. Primo, Strong regularizing effect of a gradient term in the heat equation with the Hardy potential, J. Funct. Anal. 258 (4) (2010), 1247–1272.

- [3] B. Abdellaoui, I. Peral, A. Primo, Optimal results for parabolic problems arising in some physical models with critical growth in the gradient respect to a Hardy potential, *Adv. Math.* 225 (6) (2010), 2967–3021.
- [4] B. Abdellaoui, I. Peral, A. Primo, Influence of the Hardy potential in a semilinear heat equation, *Proc. Roy. Soc. Edinburgh Sect. A* 139 (5) (2009), 897–926.
- [5] J. M. Arrieta, On boundedness of solutions of reaction-diffusion equations with nonlinear boundary conditions. *Proc. Amer. Math. Soc.* 136 (1) (2008), 151–160.
- [6] J. M. Arrieta, A. N. Carvalho, A. Rodríguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities. *J. Differential Equations* 156 (2) (1999), 376–406.
- [7] J. M. Arrieta, A. N. Carvalho, A. Rodríguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds. *Comm. Partial Differential Equations* 25 (1-2) (2000), 1–37.
- [8] P. Baras and J. Goldstein, The heat equation with a singular potential, *Trans. Amer. Math. Soc.* 284 (1984), 121–139.
- [9] C. Bennett, R. Sharpley, *Interpolation of operators*, Academic Press, Pure and Applied Mathematics 129, 1988.
- [10] J. Bergh, J. Lofstrom, *Interpolation Spaces*, Springer, Berlin-Heidelberg-New York, 1976.
- [11] X. Cabré, Y. Martel, Existence versus explosion instantanée pour des équations de la chaleur linéaires avec potentiel singulier, *C. R. Acad. Sci. Paris Sér. I Math.* 329 (11) (1999), 973–978.
- [12] M. Chaves, J. García Azorero, On bifurcation and uniqueness results for some semilinear elliptic equations involving a singular potential, *J. Eur. Math. Soc.* 8 (2) (2006), 229–242.
- [13] D. Daners, Heat kernel estimates for operators with boundary conditions. *Math. Nachr.* 217 (2000), 13–41.
- [14] D. Daners, Robin boundary value problems on arbitrary domains. *Trans. Amer. Math. Soc.* 352 (9) (2000), 4207–4236.
- [15] V. Felli, E. M. Marchini, S. Terracini, On Schrödinger operators with multipolar inverse-square potentials, *Journal of Func. Anal.* 250 (2007), 265–316.
- [16] V. Felli, E. M. Marchini, S. Terracini, On Schrödinger operators with multisingular inverse-square anisotropic potentials, *Indiana Univ. Math. J.* 58 (2009), 617–676.

- [17] V. Felli, A. Pistoia, Existence of blowing-up solutions for a nonlinear elliptic equation with Hardy potential and critical growth, *Comm. Partial Differential Equations* 31 (2006), 21–56.
- [18] L. C. F. Ferreira, C. A. A. S. Mesquita, Existence and symmetries for elliptic equations with multipolar potentials and polyharmonic operators, *Indiana Univ. Math. J.* 62 (6) (2013), 1955–1982.
- [19] L. C. F. Ferreira, M. Montenegro, A Fourier approach for nonlinear equations with singular data, *Israel J. Math.* 193 (1) (2013), 83–107.
- [20] L. C. F. Ferreira, E. J. Villamizar-Roa, A semilinear heat equation with a localized nonlinear source and non-continuous initial data. *Math. Methods Appl. Sci.* 34 (15) (2011), 1910–1919.
- [21] L. C. F. Ferreira, E. J. Villamizar-Roa, Well-posedness and asymptotic behavior for convection problem in \mathbb{R}^n , *Nonlinearity* 9 (2006) 2169–219.
- [22] W.M. Frank, D.J. Land and R.M. Spector, Singular potentials, *Rev. Modern Physics*, 43 (1971), 36–98.
- [23] J. Garcia-Azorero, I. Peral, J. D. Rossi, A convex-concave problem with a nonlinear boundary condition. *J. Differential Equations* 198 (1) (2004), 91–128.
- [24] J. A. Goldstein, Q. S. Zhang, Linear parabolic equations with strong singular potentials, *Trans. Amer. Math. Soc.* 355 (1) (2003), 197–211.
- [25] K. Ishige, M. Ishiwata, Heat equation with a singular potential on the boundary and the Kato inequality. *J. Anal. Math.* 118 (1) (2012), 161–176.
- [26] K. Ishige, T. Kawakami, Global solutions of heat equation with a nonlinear boundary condition, *Calc. Var.* **39** (2010), 429–457.
- [27] N. I. Karachalios, N. B. Zographopoulos, The semiflow of a reaction diffusion equation with a singular potential, *Manuscripta Math.* 130 (1) (2009), 63–91.
- [28] T. Kato, Strong L^p -solutions of the Navier-Stokes equation in \mathbb{R}^n , with applications to weak solutions. *Math. Z.* 187 (4) (1984), 471–480.
- [29] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. *Comm. Partial Differential Equations* 19 (5-6) (1994), 959–1014.
- [30] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics*, Pergamon Press Ltd., London, 1965.

- [31] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Research Notes in Mathematics, vol. 431, Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [32] J. M. Lévy-Leblond, Electron capture by polar molecules, *Phys. Rev.* 153 (1967), 1–4.
- [33] V. Liskevich, A. Shishkov, Z. Sobol, Singular solutions to the heat equations with nonlinear absorption and Hardy potentials, *Commun. Contemp. Math.* 14 (2) (2012), 1250013, 28 pp.
- [34] R. O’Neil, Convolution operators and $L(p, q)$ spaces, *Duke Math. J.* 30 (1963) 129–142.
- [35] I. Peral, J. L. Vázquez, On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term, *Arch. Rational Mech. Anal.* 129 (3) (1995), 201–224.
- [36] P. Quittner, W. Reichel, Very weak solutions to elliptic equations with nonlinear Neumann boundary conditions. *Calc. Var. Partial Differential Equations* 32 (4) (2008), 429–452.
- [37] P. Quittner, P. Souplet, Bounds of global solutions of parabolic problems with nonlinear boundary conditions. *Indiana Univ. Math. J.* 52 (4) (2003), 875–900.
- [38] G. Reyes, A. Tesei, Self-similar solutions of a semilinear parabolic equation with inverse-square potential, *J. Differential Equations* 219 (1) (2005), 40–77.
- [39] A. Rodríguez-Bernal, Attractors for parabolic equations with nonlinear boundary conditions, critical exponents, and singular initial data. *J. Differential Equations* 181 (1) (2002), 165–196.
- [40] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, *Trans. Amer. Math. Soc.* 357 (2005), 2909–2938.
- [41] S. Ukai, A solution formula for the Stokes equation in \mathbb{R}_+^n , *Comm. Pure Appl. Math.* **40** (5) (1987), 611–621.
- [42] J. L. Vazquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, *J. Funct. Anal.* 173 (1) (2000), 103–153.
- [43] M. Yamazaki, The Navier-Stokes equations in the weak- L^n space with time-dependent external force. *Math. Ann.* 317 (4) (2000), 635–675.